

Scaling and commensurate-incommensurate crossover for the $d = 2$, $z = 2$ quantum critical point of itinerant antiferromagnets.

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Abstract. - Quantum critical points exist at zero temperature, yet, experimentally their influence seems to extend over a large part of the phase diagram of systems such as heavy-fermion compounds and high-temperature superconductors. Theoretically, however, it is generally not known over what range of parameters the physics is governed by the quantum critical point. We answer this question for the spin-density wave to fermi-liquid quantum critical point in the two-dimensional Hubbard model. This problem is in the $d = 2, z = 2$ universality class. We use the Two-Particle Self-Consistent approach, which is accurate from weak to intermediate coupling, and whose critical behavior is the same as for the self-consistent-renormalized approach of Moriya. Despite the presence of logarithmic corrections, numerical results demonstrate that quantum critical scaling for the static magnetic susceptibility can extend up to very high temperatures but that the commensurate to incommensurate crossover leads to deviations to scaling.

There are strong indications that quantum critical points, i.e. critical points at zero temperature, influence the physical properties of materials at surprisingly high temperature. But the precise region of temperature over which this influence is felt is currently not well understood. In solvable models of quantum critical behavior, [1] power law scaling and universality associated with quantum criticality were found up to temperatures of order $J/2$ where J is the exchange constant. That is in sharp contrast with classical critical points where scaling is usually observed only in a very narrow range around the critical point. The importance of quantum critical points [2] has thus come to the fore in the study of numerous materials, including high-temperature superconductors and heavy-fermion materials where quantum phase transitions and power law scaling are observed. [3]

One particularly relevant case in this context is that of itinerant electrons undergoing a paramagnetic Fermi liquid to spin-density wave (SDW) transition in two dimensions. The Hubbard model is the simplest microscopic model that contains this physics. There is no analog of

the Ginzburg criterion that allows us to determine the parameter range where the influence of the quantum critical point is important. In that regime, temperature acts like a finite-size cutoff for the correlation length ξ . In this paper, we quantify the range of temperature where quantum critical scaling is observable in this model, in other words we find out whether details of the Fermi surface (that lead for example to commensurate-incommensurate (C-I) crossovers), logarithmic corrections, or interaction effects, lead to sizable deviations from quantum critical behavior at finite temperature.

For this problem, the dynamical critical exponent z is equal to two and the corresponding universality class ($d + z = 4$) at the upper critical dimension is ill understood. [2–4] In particular, the standard Hertz-Millis action for quantum critical phenomena is invalid. [5,6] More specifically, when the SDW is commensurate at the antiferromagnetic wave vector, it has been suggested that all the coefficients of the Ginzburg-Landau-Wilson action become singular and that the spin susceptibility scaling becomes $1/T^\eta$ with $\eta < 1$. [5]. The generic case where the

SDW is not commensurate should not have these singularities.

An alternative approach is the self-consistent renormalized theory of Moriya. This theory includes logarithmic corrections. [3, 7] However, it is not adequate to make quantitative predictions for deviations from quantum critical effects in the Hubbard model since it necessitates phenomenological constants as input. In addition, it does not satisfy the Pauli principle. In a theory that satisfies the Pauli principle, there is an interaction independent sum rule on spin and charge susceptibilities [8] that should be enforced and, in addition, the local moment, $\langle S_z^2 \rangle$, with S_z the z component of the local spin, cannot exceed $\hbar^2 n/4$ when the filling n satisfies $n < 1$ and $\hbar^2(2 - n)/4$ when $n > 1$. There is nothing that imposes these constraints in the theory of Moriya.

Method and model: We use the non-perturbative Two-Particle Self-Consistent (TPSC) approach [8]. This approach respects the Pauli principle, the Mermin-Wagner theorem and conservation laws. It also contains quantum fluctuations in crossed channels that lead to Kanamori-Brückner screening. [9] It is valid in the weak to intermediate coupling regime ($U \lesssim 6t$) and not too deep in the renormalized classical regime where a pseudogap is observed. Numerical results obtained from TPSC in its domain of validity are extremely close to the numerically exact solution obtained (barring statistical errors) with benchmark Quantum Monte Carlo calculations on the Hubbard model. [8–13]. The approach gives a satisfactory description of the pseudogap in electron-doped cuprates in a wide doping range. [14, 15] It has been shown to be in the $N = \infty$ universality class, where N is the number of components in the $O(N)$ vector model. [16] Since we are looking for deviations from universality and the theory has been benchmarked in non-universal regimes, we argue that our results are reliable for this question, even though we cannot claim to be completely accurate in the $N = 3$ regime. Nevertheless, we will demonstrate that TPSC has the same critical behavior as Moriya theory and hence has the same logarithmic corrections. These logarithms have the same functional form as those of the renormalization group asymptotically close to the quantum critical point, but in TPSC and in Moriya theory the mode-mode coupling term does not flow, hence the corrections may differ in the details from the renormalization group. [3] Quantum critical behavior of the susceptibility and of the self-energy in the closely related spin-fermion model has been discussed by Abanov et al. [17]

We study the $t - t' - U$ two-dimensional Hubbard model on the square lattice at weak to intermediate coupling,

$$H = - \sum_{\langle i,j \rangle, \sigma} t_{i,j} (c_{i,\sigma}^\dagger c_{j,\sigma} + h.c.) + U \sum_i n_{i,\uparrow} n_{i,\downarrow} \quad (1)$$

where $t_{i,j}$ are the hopping integrals, i, j are the site index, σ is the spin label, $c_{i,\sigma}^\dagger$ and $c_{i,\sigma}$ are the particle creation and annihilation operators. Each doubly occupied site costs an

energy U and $n_{i,\sigma} = c_{i,\sigma}^\dagger c_{i,\sigma}$. The units are such that $\hbar = 1$, $k_B = 1$ and lattice spacing is unity. All the numerical results are presented in units where $t = 1$. The dispersion relation is written as:

$$\epsilon_{\mathbf{k}} = -2t(\cos(k_x) + \cos(k_y)) - 4t' \cos(k_x) \cos(k_y). \quad (2)$$

We concentrate on the behavior of the spin susceptibility. In TPSC, the retarded spin susceptibility $\chi(\mathbf{q}, \omega)$ is written as:

$$\chi(\mathbf{q}, \omega) = \frac{\chi_0(\mathbf{q}, \omega)}{1 - \frac{U_{sp}}{2} \chi_0(\mathbf{q}, \omega)}, \quad (3)$$

where $\chi_0(\mathbf{q}, \omega)$ is the retarded Lindhard function at wave vector \mathbf{q} and angular frequency ω . The effective spin interaction U_{sp} is evaluated without adjustable parameter using the *ansatz* [8, 9]

$$U \langle n_\uparrow n_\downarrow \rangle = U_{sp} \langle n_\uparrow \rangle \langle n_\downarrow \rangle \quad (n < 1) \quad (4)$$

$$U \langle (1 - n_\uparrow)(1 - n_\downarrow) \rangle = U_{sp} \langle (1 - n_\uparrow) \rangle \langle (1 - n_\downarrow) \rangle \quad (n > 1) \quad (5)$$

with the local-moment sum rule that follows from the fluctuation-dissipation theorem

$$n - 2 \langle n_\uparrow n_\downarrow \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d^2 q}{(2\pi)^2} \frac{2}{1 - e^{-\omega/T}} \chi''(\mathbf{q}, \omega) \quad (6)$$

where $\chi''(\mathbf{q}, \omega) = \text{Im} \chi(\mathbf{q}, \omega)$, T is the temperature, and $\langle n_\uparrow n_\downarrow \rangle$ double occupancy. We dropped the site index using translational invariance and we used the Pauli principle to write

$$S^2 \equiv \langle (n_\uparrow - n_\downarrow)^2 \rangle = n - 2 \langle n_\uparrow n_\downarrow \rangle. \quad (7)$$

All the numerical results below are obtained using the Matsubara frequency version of equations (2) to (6) without any approximation, hence they are valid at arbitrary distance from the quantum critical point. Before proceeding, we show however that the quantum critical behavior of TPSC is the same as that of the self-consistent renormalized theory of Moriya and we discuss conditions for scaling.

Analytical results near the quantum critical point. When the correlation length is large, one can expand the denominator of the TPSC spin susceptibility around the wave vectors \mathbf{q}_i where the maxima in χ_0 occur to obtain

$$\chi''(\mathbf{q}, \omega) = \frac{2}{U_{sp} \xi_0^2} \sum_i \frac{\omega/\Gamma_0}{(\xi^{-2} + (\mathbf{q} - \mathbf{q}_i)^2)^2 + (\omega/\Gamma_0)^2}. \quad (8)$$

Defining $U_{mf} = 2/\chi_0(\mathbf{q}_i, 0)$ as the value of the interaction at the mean-field SDW transition, the other quantities in the previous expression are

$$\xi^2 \equiv \xi_0^2 \left(\frac{U_{sp}}{\delta U} \right), \quad (9)$$

$$\delta U \equiv U_{mf} - U_{sp}, \quad (10)$$

$$\xi_0^2 \equiv - \frac{1}{2\chi_0(0, \mathbf{q}_i)} \left. \frac{\partial^2 \chi_0(\mathbf{q}, 0)}{\partial q^2} \right|_{\mathbf{q}_i}, \quad (11)$$

$$\frac{1}{\Gamma_0} \equiv \frac{1}{\xi_0^2 \chi_0(\mathbf{q}_i, 0)} \left. \frac{\partial \chi_0^R(\mathbf{q}_i, \omega)}{\partial (i\omega)} \right|_{\omega=0}. \quad (12)$$

In the expression for the spin susceptibility, the denominators are expanded around each of the four incommensurate wave vectors, or only around the (π, π) wave vector depending on the situation. We checked explicitly that higher powers of $(\mathbf{q} - \mathbf{q}_i)$ do not improve the description of the C-I crossover and are not relevant.

To determine the quantum critical behavior, one subtracts the self-consistency condition Eq.(6) for a value of temperature and filling close to the quantum critical point from the same equation evaluated at that critical point

$$S^2 - S_c^2 = \int_0^\infty \frac{d\omega}{\pi} \int \frac{d^2 q}{(2\pi)^2} \left[\frac{2}{(e^{\omega/T} - 1)} \chi''(\mathbf{q}, \omega) + \chi''(\mathbf{q}, \omega) - \chi_c''(\mathbf{q}, \omega) \right]. \quad (13)$$

In the above expression $\chi_c''(\mathbf{q}, \omega)$ is evaluated at the quantum critical point where $\xi^{-2} = 0$. (From now on, a subscript c means that the quantity is evaluated at the quantum critical point.) One then performs the integrals over momentum in a circular domain with cutoff q_B and then the frequency integrals. To write the final answer, it is useful to follow Moriya et al. [18] and to define

$$T_0 = \frac{\Gamma_0 q_B^2}{2\pi}. \quad (14)$$

and dimensionless measures of ξ and T

$$y \equiv \frac{\xi^{-2}}{q_B^2}; \quad \tau \equiv \frac{T}{T_0}. \quad (15)$$

The definition of Γ_0 Eq.(12) and the fact that ξ_0 and q_B^{-1} are both of the order of the lattice spacing shows that T_0 is a temperature of the order of the Fermi energy. With these definitions and a single maximum in the susceptibility, the self-consistency expression takes the form

$$y(1 - \ln y) = y_0 + \frac{\tau}{\pi} \left[\phi\left(\frac{y}{\tau}\right) - \phi\left(\frac{y}{\tau} + \frac{1}{\tau}\right) \right] \quad (16)$$

where terms of order y^2 have been neglected on the left-hand side. We defined

$$y_0 \equiv -\frac{U_{sp}\xi_0^2}{T_0} (S^2 - S_c^2) \quad (17)$$

and obtained $\phi(x)$ from the second Binet log gamma formula [19]

$$\begin{aligned} \phi(x) &= 2 \int_0^\infty dz \frac{1}{e^{2\pi z} - 1} \arctan\left(\frac{z}{x}\right) \\ &= \ln \Gamma(x) - \left(x - \frac{1}{2}\right) \ln x + x - \frac{1}{2} \ln(2\pi) \end{aligned} \quad (18)$$

with $\text{Re}[z] > 0$ and $\Gamma(x)$ Euler's gamma function. The quantity y_0 in Eq.(17) measures the deviation from the quantum critical point. Apart from the logarithm, the self-consistency relation Eq.(16) has the same functional

form as Eq.(2.8) in Ref. [18] Logarithmic corrections for that theory are mentioned without proof in Ref. [7].

For large local moment, $S^2 > S_c^2$, there is an SDW ordered ground state and $y_0 < 0$. The case $y_0 > 0$ corresponds to the Fermi liquid ground state and $y_0 = 0$ to the quantum critical point. The full filling and temperature dependence of y_0 is found from the definitions of U_{sp} and ξ^{-2} . For example in the hole doped case, defining $\Delta n \equiv n - n_c$, we have

$$y_0 = -\frac{U_{mf}\xi_0^2}{T_0} \frac{1}{\frac{\xi^{-2}}{\xi_0^2} + 1} \left[\Delta n - \frac{U_{mf}}{2U} \frac{n^2}{\frac{\xi^{-2}}{\xi_0^2} + 1} + \frac{U_{c,mf}}{2U} n_c^2 \right]. \quad (20)$$

Thus, y_0 depends on $y = \xi^{-2}/q_B^2$ but in the critical regime $y \ll 1$ and $y_0 \ll 1$ so we can neglect terms of order yy_0 . The quantity y_0 can then be written in the form

$$y_0 = -(a\Delta n + bT) \quad (21)$$

where a is a positive number.

The various limiting solutions for the critical behavior of the dimensionless correlation length can be obtained from the self-consistency condition Eq.(16) as follows. [18] For $y_0 < 0$, one must take the limit $y \rightarrow 0$ first, then $\phi\left(\frac{y}{\tau}\right) - \phi\left(\frac{y}{\tau} + \frac{1}{\tau}\right) \simeq -\frac{1}{2} \ln\left(\frac{2\pi y}{\tau}\right)$ and since y is exponentially small, $y - y \ln y$ can be neglected on the left-hand side leading to $y \simeq (\tau \exp(2\pi y_0/\tau))/2\pi$. This is the renormalized classical regime where the correlation length grows exponentially. At the quantum critical point $y_0 = 0$, the same limit of the ϕ functions applies and one must find the solution of $-y \ln y \simeq -\frac{\tau}{2\pi} \ln\left(\frac{2\pi y}{\tau}\right)$ which is approximately $y \sim \tau \ln(|\ln \tau|)/|\ln \tau|$, as in the renormalization group. [7] Finally, in the Fermi liquid regime, $y_0 > 0$, the correlation length (and hence y) is finite so the $\tau \rightarrow 0$ limit must be taken first and $\phi\left(\frac{y}{\tau}\right) - \phi\left(\frac{y}{\tau} + \frac{1}{\tau}\right) \simeq \tau/(12y)$ which yields $y \sim y_0 + O(\tau^2)$. At $\tau = 0$ on the Fermi liquid side, there are logarithmic corrections to the dependence of y on y_0 asymptotically close to the quantum critical point since $-y \ln y \simeq y_0$ whose approximate solution is $y \simeq -y_0/\ln y_0$.

In all regimes where $\ln y$ in the self-consistency Eq.(16) can be neglected (large T) or replaced by a constant in the temperature range of interest, one can write

$$\frac{y}{\tau} \equiv F\left(\frac{\Delta n}{\tau}, \frac{1}{\tau}\right) \quad (22)$$

where the scaling function F is the solution of

$$cF = \frac{y_0}{\tau} + \frac{1}{\pi} \left[\phi(F) - \phi\left(F + \frac{1}{\tau}\right) \right]. \quad (23)$$

with $c = 1 - \ln y_t$, y_t being the typical value of y in the range of temperature under study. We have already discussed limiting cases of F above. We demonstrate numerically below that in the range $0.01t < T < t$ logarithmic corrections are negligible so that scaling holds to an excellent approximation, except at the C-I crossover.

Scaling function. When the explored temperature range is limited on a logarithmic scale, or when T is large, logarithmic corrections can be neglected. In addition, in the limit where τ is much smaller than y/τ , the scaling function F in Eq.(22) depends only on $\Delta n/\tau$ since we are in the limiting case $\phi(F + \frac{1}{\tau}) \rightarrow \phi(\infty) = 0$ in the equation that defines F , Eq.(23). This case occurs when the ground state is paramagnetic, $y_0 > 0$, or above the crossover line to the renormalized classical regime that occurs when $y_0 < 0$. In such cases, near anyone of the maxima located at \mathbf{q}_i , the quantity $y = \xi^{-2}/q_B^2$ scales as $\tau F(\Delta n/\tau, \infty)$ so the spin susceptibility Eq.(8) as a function of an arbitrary scale factor s obeys the scaling relation

$$\begin{aligned} & \chi(T, \Delta n, |\mathbf{q} - \mathbf{q}_i|, \omega) - R \\ &= s^{\gamma/\nu} \chi_1(s^{1/\nu} T, s^{\phi/\nu} \Delta n, s|\mathbf{q} - \mathbf{q}_i|, s^z \omega) \end{aligned} \quad (24)$$

where the exponents have values $\gamma = 1$, $\nu = 1/2$, $z = 2$ and $\phi = 1$. In the above equation, R will not be important only if the incommensurate peaks are much narrower in momentum space than the inverse correlation length. Let $\omega = 0$ for now and drop the dependence on that variable. Following the above discussions on the behavior of the correlation length, the susceptibility χ_1 on the right-hand side of the last equation should be, within log corrections, a universal function of its arguments but with the overall scale of each argument and of χ_1 non-universal. Setting, $q \equiv |\mathbf{q} - \mathbf{q}_i| = 0$, $\omega = 0$ and choosing s such that $T s^{1/\nu} = 1$ we find,

$$\chi(T, \Delta n, 0, 0) = \frac{1}{T} X\left(\frac{\Delta n}{T}\right) + R. \quad (25)$$

where the scale of the function X defined by this equation and an overall prefactor in front of the argument are not universal. X is the quantity we will focus on, but we note in passing that the general form Eq.(24) with the given exponents implies ω/T scaling for the q integrated susceptibility. [20] Non-universal factors such as U_{sp} , ξ_0 and Γ_0 that enter the spin susceptibility can have some temperature and filling dependence in TPSC that can in principle lead to deviations to scaling. In renormalization group language, these dependencies are the irrelevant variables whose importance we are trying to gauge to delimit the scaling regime.

Commensurate-incommensurate crossover. In a strict sense, the value of \mathbf{q} should be fixed at $\mathbf{q} = \mathbf{q}_i$ ($T = 0$) to check quantum critical scaling. However, \mathbf{q}_i itself depends on temperature in general. At high temperature \mathbf{q}_i equals $\mathbf{Q} = (\pi, \pi)$, becoming incommensurate at low temperature. The susceptibility there shows four symmetry related peaks for the model we consider. [21] The value of $\mathbf{q}_i(T)$ clearly depends on details of the Fermi surface and is thus non-universal. The above scaling form Eq.(24) nevertheless suggests that scaling in the $T, \Delta n$ plane as in Eq.(25) should occur when $\mathbf{q} = \mathbf{q}_i(T)$. It is not however possible to define $\mathbf{q}_i(T)$ in the C-I crossover regime.

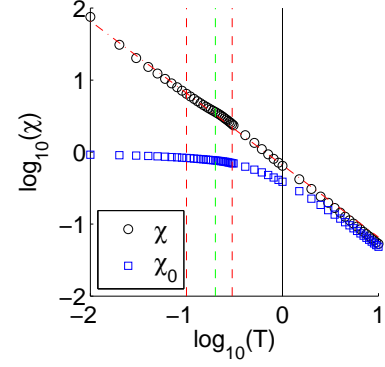


Fig. 1: (Color online) $\log_{10}(\chi)$ as a function of $\log_{10}(T)$ for $U = 4t$ and $t' = 0$ (open circles) at $n_c = 0.84$ and for the corresponding non-interacting susceptibility at $U = 0$ (open squares). The vertical dashed lines indicate the commensurate to incommensurate crossover region. The red dash-dotted line has a slope -1 . The black vertical line simply indicates $T = t$.

In that regime, incommensurate peaks necessarily overlap since the second derivative of χ vanishes at $\mathbf{q} = \mathbf{Q}$ when the crossover begins, reflecting the fact that there is a broad maximum at \mathbf{Q} that is splitting into four overlapping peaks. R in the general scaling function Eq.(25) is not negligible in the C-I crossover region. On general grounds then, we expect deviations to scaling there. One may think that a better strategy to prove scaling is to measure the correlation length ξ as a function of T and Δn , but ξ cannot be determined in the C-I crossover regime for the same above reasons.

From now on, we thus look for scaling with the susceptibility evaluated at its maximum, $\chi(T, \Delta n, |\mathbf{q}_{\max} - \mathbf{q}_i|, 0)$. This is a well defined quantity experimentally and far from the C-I crossover we will have $\mathbf{q}_{\max} = \mathbf{q}_i(T)$.

Numerical results: Let us first verify the scaling at the quantum critical point $\Delta n = 0$. Fig. (1) shows a log-log plot of both the interacting (open circles) and non-interacting (open squares) susceptibilities as a function of temperature for two different sets of parameters.

For temperatures larger than hopping t , one obtains trivial $1/T$ scaling for both the interacting and non-interacting susceptibilities. While the non-interacting susceptibility flattens at lower temperature, the interacting susceptibility shows quantum critical $1/T$ scaling down to the lowest temperature we could reach, namely $T = 0.01t$. We will see that the $1/T$ scaling at $T > 1$ that comes from the non-interacting susceptibility does not obey the scaling equation Eq.(25). It is also clear from Fig. (1) that deviations to scaling occur in the C-I crossover regime delimited by the vertical red lines. It is remarkable however that the same straight line fits both the commensurate and the incommensurate regimes. This suggests that non-universal scale factors are very similar on either sides of the commensurate-incommensurate transition. The slight upward curvature at the lowest temperatures is not inconsistent with effects of logarithmic corrections.

To verify the full scaling Eq.(25), we plot $T\chi$ as a function of $|\Delta n|/T$ on a log-log plot in Fig. 2. We take values of n on the Fermi liquid side of n_c . For a given band structure and interaction, it is only when one has found the correct values of the critical $n = n_c$ that all the curves for different fillings and temperature collapse on the same curve. We found, when $t' = 0$, that $n_c = 0.926, 0.840$ and 0.795 for $U = 2t, 4t$ and $6t$ respectively and $n_c = 1.180$ for the electron-doped case with $U = 6t, t' = -0.05t$. More values can be found in the thesis which is the basis for all the results of the present paper. [20]

The straight line of slope -1 at large $\Delta n/T$ in Figs. 2a,c,d. corresponds to the Fermi liquid regime where both the susceptibility and the correlation length are temperature independent, but diverge as one approaches the quantum critical point. In that regime, χ scales as $\xi^2 \sim 1/\Delta n$ when logarithmic corrections are negligible. The $1/T$ scaling of χ corresponds to plateaus on the left of Figs. 2a,c,d. The deviations from a plateau come from the C-I crossover. To show that the scaling is non-trivial, in Fig. 2b we do not multiply the susceptibility by T on the vertical axis. The lined up circles that can be caught by the eye correspond to different temperatures for a given filling n , the fillings closest to n_c being to the left.

Scale factors depending on band structure and interaction strength should not influence the shape of the scaling function. A simple translation in the $T - \Delta n$ plane of the curves for different parameters should allow all of them to collapse. In Fig. 2d, we show scaling functions for various parameters but without translation for non-universal factors. One sees that if there were no deviations to scaling associated with the C-I crossover in the plateau region, simple translation would make all the curves nearly collapse. This also shows that logarithms do not have a large influence on scaling in this temperature range.

In Fig. 3, data analogous to those in Fig. 2 are represented by black open circles and are filtered out near the C-I crossover. The missing data is particularly clear in Fig. 3b where we do not scale the vertical axis. If T_i is the temperature where the crossover occurs for a given doping, the data were filtered in the range $T_i - \Delta T < T < T_i + \Delta T$ ($\Delta T \sim 0.2t$) for densities $n_c - n < 0.04$. For larger values of $n_c - n$, the data is sufficiently far from the C-I crossover that no filtering is required. The remaining data are those beyond the C-I crossover both above (commensurate) and below (incommensurate) T_i . One sees that a plateau is recovered (black open circles) for all three values of the interaction strength appearing in Figs. 3a,c,d as expected in the quantum critical regime.

We now turn to the high temperature limit of the quantum critical scaling. While the black open circles in Fig. 3 are for $T < t$, those for $t < T < 10t$ are represented by red crosses. The deviations to scaling for $t < T < 10t$ are obvious. Even though the non-interacting susceptibility scales as $1/T$ for $T > t$ as we saw in Fig. 1, it does not pollute the scaling associated purely with the quantum critical point. The latter occurs for $T < t$, with the

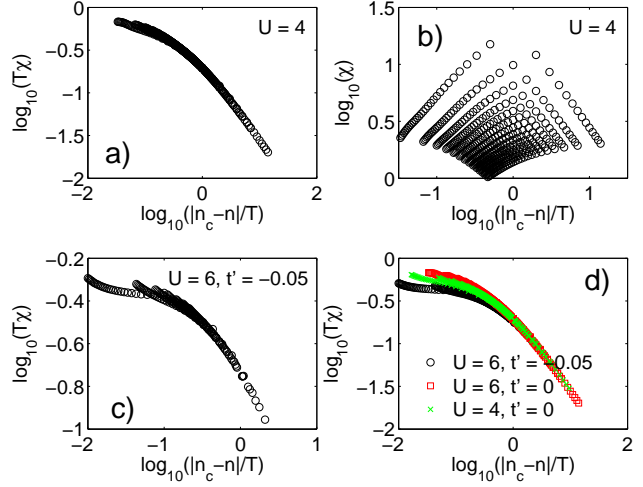


Fig. 2: (Color online) a) $\log_{10}(T\chi)$ as a function of $\log_{10}\left(\frac{|\Delta n|}{T}\right)$ for $U = 4t$ and $t' = 0$. b) Shows the data of figure a) with the susceptibility unscaled $\log_{10}(\chi)$ c) $\log_{10}(T\chi)$ as a function of $\log_{10}\left(\frac{|\Delta n|}{T}\right)$ for $U = 6t$ and $t' = -0.05t$. d) compares the scaled data $\log_{10}(T\chi)$ as a function of $\log_{10}\left(\frac{|\Delta n|}{T}\right)$ for $U = 4t$ and $t' = 0$ (green crosses), $U = 6t$ and $t' = 0$ (red open squares) and $U = 6t$ and $t' = -0.05t$ (black open circles). On figures a) and c) we can see deviations from the scaling at $\log_{10}\left(\frac{|\Delta n|}{T}\right) < -1$ that are due to the commensurate incommensurate crossover (see text). In these figures, for $t' = 0$ we considered hole doping with $n_c - n > 0$, the smallest n being $n = 0.7$. For $t' = -0.05t$, we took electron doping with the largest n equal to $n = 1.3$ and $n - n_c$ as argument of the log.

caveat concerning the C-I crossover. The maximum T for scaling, $T \sim t$, is an important result that applies in the weak to intermediate coupling regime we have considered here. Clearly quantum critical behavior must disappear at $U = 0$, so there should be some U dependence to the upper temperature cutoff. At the intermediate coupling values that we considered, the temperature range over which quantum critical scaling is observed should be compared to what would have been naively estimated by substituting $U = 4t$ and $U = 6t$ in $J = 4t^2/U$ [1], obtaining respectively $J/2 = t/2$ and $J/2 = t/3$. Basically, the upper limit of $T \sim t$ is essentially the degeneracy temperature for Fermi-Dirac statistics, which is of the same order as T_0 . The irrelevant temperature dependencies of all quantities are thus on this scale.

Conclusion. The quantum critical behavior of TPSC for the $d = 2, z = 2$ universality class is the same as that of the self-consistent renormalized theory of Moriya, hence it includes logarithmic corrections. In TPSC there is no adjustable parameter. By explicit numerical calculations away from the renormalized classical regime of the $d = 2$ Hubbard model in the weak to intermediate coupling, we have been able to show that logarithmic corrections are not really apparent in the range of temperature $0.01 < T < t$ and that the maximum static spin susceptibility in the (T, n) plane obeys quantum critical scaling.

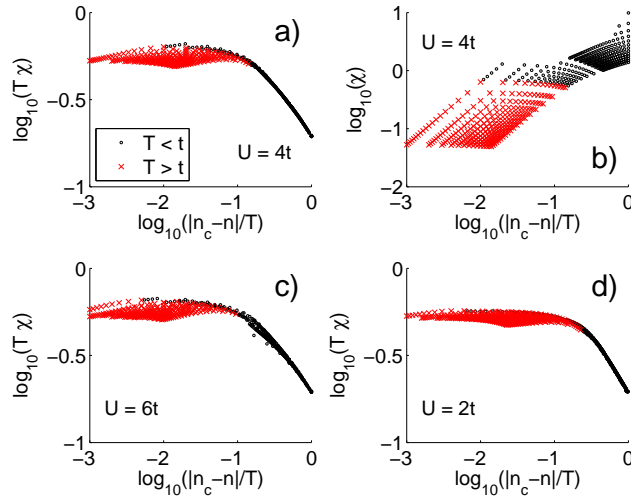


Fig. 3: (Color online) The black circles are the data for $T < t$ and the red crosses for $T > t$. a) Scaled data $\log_{10}(T\chi)$ for $U = 4t$. b) Unscaled values $\log_{10}(\chi)$ for the same U . c) $\log_{10}(T\chi)$ for $U = 6t$ d) for $U = 2t$. All four panels are at $t' = 0$. The high temperature limit of the quantum critical scaling is of the order $T \sim t$.

ing. However, near the commensurate-incommensurate crossover, one finds obvious non-universal temperature and filling dependence. Everywhere else, the (T, n) dependence of the non-universal scale factors is relatively weak. Strong deviations from scaling occur at temperatures of order t , the degeneracy temperature, reflecting the fact that the temperature dependence of most irrelevant terms is on the scale of the Fermi energy. That high temperature limit should be contrasted with $J/2$ found in the strong coupling case. [1] In generic cases the upper limit $T \sim t$ is well above room temperature. In experiment however, the non-universality due to the C-I crossover may make the identification of quantum critical scaling difficult. And since the (T, n) dependence of \mathbf{q}_i is non universal, one may encounter cases where this is in practice impossible.

Electron-doped high-temperature superconductors appear as an ideal system to check quantum critical scaling since they seem well described by the $d = 2$ one-band Hubbard model at weak to intermediate coupling. [14, 15] And experiments [22, 23] strongly suggest the presence of a quantum critical point in these materials. In the case of heavy fermions there are examples of SDW-Fermi liquid quantum critical behavior. [3] However, these are multi-band systems where there are additional energy scales, such as the Kondo coherence scale, so our results would apply only in regimes where an effective one-band Hubbard model applies.

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